Optimal codes

A tree code is called *optimal* (for a given probability distribution) if no other code with a lower mean codeword length exists.

There are of course several codes with the same mean codeword length. The simplest example is to just switch all ones to zeros and all zeros to ones in the codewords.

You can always switch the places of two nodes of the code tree at the same depth and still have a code with the same mean codeword length, since all codewords will still have the same lengths.

Even codes with different sets of codeword lengths can have the same mean codeword length.

Upper bound for optimal codes

Given that we code one symbol at a time, an optimal code satisfies $\bar{l} < H(X_j) + 1$

Let $l_i = \lceil -\log p_i \rceil$. We have that $-\log p_i \leq \lceil -\log p_i \rceil < -\log p_i + 1$.

$$\begin{aligned} \sum_{i=1}^{L} 2^{-l_i} &= \sum_{i=1}^{L} 2^{-\lceil -\log p_i \rceil} \\ &\leq \sum_{i=1}^{L} 2^{\log p_i} \\ &= \sum_{i=1}^{L} p_i \; = \; 1 \end{aligned}$$

Kraft's inequality is satisfied, therefore a tree code with the given codeword lengths exists.

Upper bound for optimal codes, cont.

What's the mean codeword length of this code?

$$\begin{aligned}
\bar{l} &= \sum_{i=1}^{L} p_i \cdot l_i = \sum_{i=1}^{L} p_i \cdot \lceil -\log p_i \rceil \\
&< \sum_{i=1}^{L} p_i \cdot (-\log p_i + 1) \\
&= -\sum_{i=1}^{L} p_i \cdot \log p_i + \sum_{i=1}^{L} p_i = H(X_j) + 1
\end{aligned}$$

An optimal code can't be worse than this code, then it wouldn't be optimal. Thus, the mean codeword length for an optimal code also satisfies $\bar{l} < H(X_j) + 1$.

NOTE: If $p_i = 2^{-k_i}, \forall i$ for integers k_i, we can construct a code with codeword lengths k_i and $\bar{l} = H(X_j)$.

Bounds

We have showed that for a memoryless source where we code one symbol at a time there exists prefix codes that satisfy

$$H(X_j) \leq R = \bar{l} < H(X_j) + 1$$

The result can easily be generalized to sources with memory where we code n symbols at a time. We then get

$$H(X_j, X_{j+1}, \ldots, X_{j+n-1}) \leq \bar{l} < H(X_j, X_{j+1}, \ldots, X_{j+n-1}) + 1$$

$$R = \frac{\bar{l}}{n}$$

$$\frac{1}{n} \cdot H(X_j, X_{j+1}, \ldots, X_{j+n-1}) \leq R < \frac{1}{n} \cdot H(X_j, X_{j+1}, \ldots, X_{j+n-1}) + \frac{1}{n}$$

By coding multiple symbols with each codeword we can get arbitrarily close to the entropy limit, both while coding sources with memory and when coding memoryless sources.

Necessary conditions

Assume that we code one symbol from the alphabet $\mathcal{A} = \{a_1, \ldots, a_L\}$ with each codeword, and that the codeword lengths are $l_1, \ldots, l_L$. Necessary conditions for the code to be optimal are

1. If $p(a_i) \leq p(a_j)$ then $l_i \geq l_j$.
2. The two least probable symbols have codewords of the same length.
3. In the code tree of an optimal code there must be two branches from each inner node.
4. Suppose that we change an inner node in the tree to a leaf by combining all leaves descending from it to a single symbol in a reduced alphabet. If the original tree was optimal for the original alphabet then the reduced tree is optimal for the reduced alphabet.

Necessary conditions, cont.

1. If not, we could switch codewords between the two symbols and get a code with lower mean codeword length.
2. Suppose we have a prefix code where the two least probable symbols have different codeword lengths. We can then create a new code by removing the last bits in the longer codeword so that the two codewords have the same length. The new set of codewords is still a prefix code, since according to 1 there are no codewords that are longer. The new code has a lower mean codeword length, and thus the original code is not optimal.
3. Suppose that a prefix code has an inner node with only one branch. We can then remove that branch and move up the subtree underneath it one level. This new code is still a prefix code, and it has a lower mean codword length. Thus the original code can not be optimal.
4. If the reduced code wasn't optimal we could construct a new code for the reduced alphabet and then expand the reduced symbol again so that we get a new code with lower mean codeword length than the original code.

Huffman coding

A simple method for constructing optimal tree codes.

Start with symbols as leaves.

In each step connect the two least probable nodes to an inner node. The probability for the new node is the sum of the probabilities of the two original nodes. If there are several nodes with the same probability to choose from it doesn't matter which ones we choose.

When we have constructed the whole code tree, we create the codewords by setting 0 and 1 on the branches in each node. Which branch that is set to 0 and which that is set to 1 doesn't matter.

Huffman codes

Assume that the most probable symbol for a memoryless source X_k has the probability $p_{\max}$. It can be shown that the mean codeword length for a Huffman code satisfies

$$\bar{l} < \begin{cases} H(X_k) + p_{\max} & ; \; p_{\max} \geq 0.5 \\ H(X_k) + p_{\max} + 0.086 & ; \; p_{\max} < 0.5 \end{cases}$$

Compare this to our earlier upper bound

$$\bar{l} < H(X_k) + 1$$

Extended Huffman codes

For small alphabets with skewed distributions, or for sources with memory, a Huffman code can be relatively far from the entropy limit. This can often be improved by *extending* the source, ie by coding several symbols at a time with each codeword.

The maximum *redundancy* (the difference between the data rate and the entropy) decreases as $\frac{1}{n}$ when we code n symbols at a time.

Note that extension doesn't guarantee that the data rate decreases, just that the upper bound comes closer to the lower bound.

Extra information

Normally we have to transmit the Huffman tree to the receiver, which will require extra data.

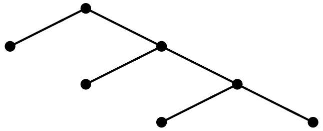
So far we have assumed that we code such a long sequence from the source that the cost for the Huffman tree can be neglected. In practical applications this is not always the case.

Straightforward method: For each symbol in the alphabet we first send the codeword length and then the actual codeword. With an alphabet of size L we can never have a Huffman codeword of length longer than $L-1$. We will thus need $L \cdot \lceil \log(L-1) \rceil + \sum_i l_i$ extra bits.

Smarter method: Just find the codeword lengths l_i using the Huffman algorithm. Given these lengths we construct a new tree code. The decoder can use the same method to construct a tree code, which means we will only have to transmit the codeword length for each symbol, which requires $L \cdot \lceil \log(L-1) \rceil$ extra bits.

Huffman tree from codeword lengths

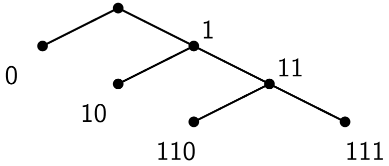
When constructing a code tree from codeword lengths we always have to assign the codewords in increasing length order. Typically the codewords are assigned in lexicographic order, so that the first codeword consists of all zeroes (or all ones).
Consider a Huffman tree where the codewords have been assigned according to this principle. For instance if we have four symbols and the codeword lengths 1, 2, 3 and 3, the codewords are 0, 10, 110, 111 which corresponds to the tree



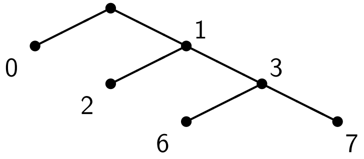
if we let the 0-branches go left and the 1-branches go right.

Huffman tree from codeword lengths

If we denote each node with the path there from the root



or, if we write the binary numbers in decimal



Huffman tree from codeword lengths

Taking a step to the right in the tree at the same depth gives that the value of the node increases by one. Taking a step downwards along the 0-branch doubles the value.

An algorithm that assigns codewords given codeword lengths can be described like this:

Assign the codewords in increasing length order. We start by giving the shortest codeword the value 0. This means we put this codeword furthest to the left at the depth of the tree corresponding to the codeword length. For each new codeword we take a step to the right at the same depth of the tree, ie we increase the value by one. If the new codeword has the same length as the previous one, the codeword is the new value. Otherwise, move down along the 0-branch until we get to the correct depth. Each step downwards corresponds to a multiplication of the value by 2. Repeat until we have assigned all codewords.

Huffman tree from codeword lengths

In pseudo code, given sorted codeword lengths length[ ] = $\{l_0 \;\; l_1 \;\; l_2 \ldots l_{L-1}\}$

```
c = 0;
code[0] = c;
for i=1 to L-1
  c = c+1;
  c = c*2^(length[i]-length[i-1]);
  code[i] = c;
end
```

The codeword for symbol k is the number code[k] written as a binary number with length[k] bits.
Note that the algorithm uses sorted lengths. We also have to keep track of which symbol that corresponds to each codeword.

Huffman tree from codeword lengths

Alternatively, if we prefer to assign the codewords in reverse lexicographic order (the shortest codeword is all ones and the longest codeword is all zeros), we can use the following pseudo code.

```
c = 2^length[0]-1;
code[0] = c;
for i=1 to L-1
  c = c*2^(length[i]-length[i-1]);
  c = c-1;
  code[i] = c;
end
```

The codeword for symbol k is the number code[k] written as a binary number with length[k] bits.