# Solutions for chapter 2-12 in Sayood

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## Chapter 2

### Problem 1

Since $0 \leq p_i \leq 1$, $p_i \cdot \log_2 p_i \leq 0$ which means that $H(X) = -\sum p_i \cdot \log_2 p_i \geq 0$. For the other inequality we consider $H(X) - \log_2 M$

$$\begin{aligned}
H(X) - \log M &= -\sum_{i=1}^{M} p_i \log p_i - \log M \\
&= -\sum_{i=1}^{M} p_i \log p_i - \sum_{i=1}^{M} p_i \log M \\
&= \sum_{i=1}^{M} p_i \log \frac{1}{M \cdot p_i} \\
&\leq \frac{1}{\ln 2} \sum_{i=1}^{M} p_i \left(\frac{1}{M \cdot p_i} - 1\right) \\
&= \frac{1}{\ln 2} \left(\sum_{i=1}^{M} \frac{1}{M} - \sum_{i=1}^{M} p_i\right) \\
&= \frac{1}{\ln 2}(1 - 1) = 0
\end{aligned}$$

where we used the fact that $\ln x \leq x - 1$ (show this!).

### Problem 3

(a) $H(X) = 2$ bits

(b) $H(X) = 1.75$ bits

(c) $H(X) \approx 1.7398$ bits

### Problem 7

(a) Not uniquely decodable

(b) Not uniquely decodable

(c) Uniquely decodable

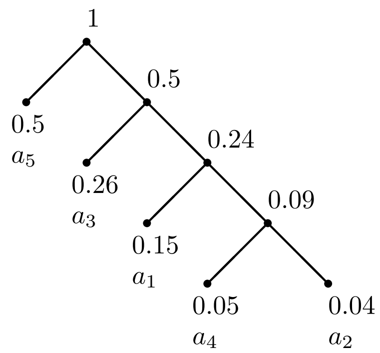
(d) Not uniquely decodable

# Chapter 3

## Problem 4

(a) $H = -\sum_{i=1}^{5} P(a_i) \cdot \log_2 P(a_i) \approx 1.8177$ bits

(b) The code tree will look like




The codewords can for example be:

| | |
|---|---|
| $a_1$ | 110 |
| $a_2$ | 1111 |
| $a_3$ | 10 |
| $a_4$ | 1110 |
| $a_5$ | 0 |

(c) The average codeword length will be

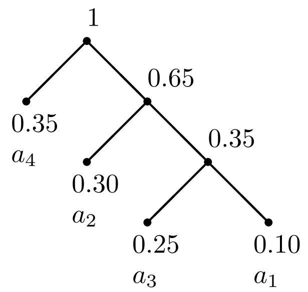
$$\bar{l} = 1 + 0.5 + 0.24 + 0.09 = 1.83 \quad \text{bits/codeword}$$

and the redundancy is thus

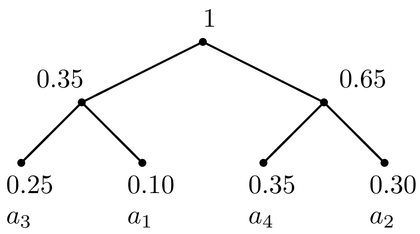
$$\bar{l} - H \approx 0.0123$$

## Problem 5

(a) The code tree will look like



(b) The code tree will look like




Both codes have the same average rate (2 bits/symbol). Since the second code has codewords of the same length, it might be more useful in an environment with errors or where buffer control is needed.

## Problem 13

The code will look similar to

| Sequence | codeword |
|---|---|
| $a_1a_1a_1$ | 000 |
| $a_1a_1a_2$ | 001 |
| $a_1a_1a_3$ | 010 |
| $a_1a_2$ | 011 |
| $a_1a_3$ | 100 |
| $a_2$ | 101 |
| $a_3$ | 110 |

and have an average rate of

$$R = \frac{3}{2.19} \approx 1.3699 \quad \text{bits/symbol}$$

(The entropy of the source is approximately 1.1568 bits/symbol.)

# Chapter 4

## Problem 5

Cumulative probability function

$$F(0) = 0, \quad F(1) = 0.2, \quad F(2) = 0.5, \quad F(3) = 1$$

The first symbol is $a_1$

$$\begin{aligned} l^{(1)} &= 0 + (1-0)\cdot 0 = 0 \\ u^{(1)} &= 0 + (1-0)\cdot 0.2 = 0.2 \end{aligned}$$

The second symbol is $a_1$

$$\begin{aligned} l^{(2)} &= 0 + (0.2-0)\cdot 0 = 0 \\ u^{(2)} &= 0 + (0.2-0)\cdot 0.2 = 0.04 \end{aligned}$$

The third symbol is $a_3$

$$\begin{aligned} l^{(3)} &= 0 + (0.04-0)\cdot 0.5 = 0.02 \\ u^{(3)} &= 0 + (0.04-0)\cdot 1 = 0.04 \end{aligned}$$

The fourth symbol is $a_2$

$$\begin{aligned} l^{(4)} &= 0.02 + (0.04-0.02)\cdot 0.2 = 0.024 \\ u^{(4)} &= 0.02 + (0.04-0.02)\cdot 0.5 = 0.03 \end{aligned}$$

The fifth symbol is $a_3$

$$\begin{aligned} l^{(5)} &= 0.024 + (0.03-0.024)\cdot 0.5 = 0.027 \\ u^{(5)} &= 0.024 + (0.03-0.024)\cdot 1 = 0.03 \end{aligned}$$

The sixth symbol is $a_1$

$$\begin{aligned} l^{(6)} &= 0.027 + (0.03-0.027)\cdot 0 = 0.027 \\ u^{(6)} &= 0.027 + (0.03-0.027)\cdot 0.2 = 0.0276 \end{aligned}$$

The tag should be a number in the interval $[0.027,\ 0.0276)$, for instance we can choose the midpoint 0.0273.

## Problem 6

The decoded sequence is

$$a_3a_2a_2a_1a_2a_1a_3a_2a_2a_3$$

# Chapter 5

In these solutions, the symbol _ is used to denote the space character.

## Problem 3

| index | string | index | string | index | string | index | string |
|---|---|---|---|---|---|---|---|
| 1 | a | 7 | _b | 13 | ra | 19 | rr |
| 2 | b | 8 | ba | 14 | ay | 20 | ray |
| 3 | r | 9 | ar | 15 | y_ | 21 | ya |
| 4 | y | 10 | r_ | 16 | _by | 22 | ar_ |
| 5 | _ | 11 | _a | 17 | y_b | 23 | _ba |
| 6 | a_ | 12 | arr | 18 | bar | 24 | |

The index sequence is

$$1, 5, 2, 1, 3, 5, 9, 3, 1, 4, 7, 15, 8, 3, 13, 4, 9, 7, 14$$

## Problem 4

| index | string | index | string | index | string | index | string |
|---|---|---|---|---|---|---|---|
| 1 | a | 8 | hi | 15 | _i | 22 | t_is |
| 2 | _ | 9 | is | 16 | is_ | 23 | s_h |
| 3 | h | 10 | s_ | 17 | _hi | 24 | his |
| 4 | i | 11 | _h | 18 | is_h | 25 | s_ha |
| 5 | s | 12 | ha | 19 | hat | 26 | at? |
| 6 | t | 13 | at | 20 | t_i | 27 | |
| 7 | th | 14 | t_ | 21 | it | 28 | |

The decoded sequence is: *this_hat_is_his_hat_it_is_his_hat*

## Problem 5

| index | string | index | string | index | string | index | string |
|---|---|---|---|---|---|---|---|
| 1 | a | 6 | at | 11 | _a | 16 | at_ |
| 2 | _ | 7 | ta | 12 | a_ | 17 | _a_ |
| 3 | r | 8 | ata | 13 | _r | 18 | _ra |
| 4 | t | 9 | atat | 14 | rat | 19 | at? |
| 5 | ra | 10 | t_ | 15 | t_a | 20 | |

The decoded sequence is: *ratatatat_a_rat_at_a_rat*

## Problem 6

The resulting sequence of triples is:
$< 0, 0, 2 >$ $< 0, 0, 1 >$ $< 0, 0, 4 >$ $< 1, 1, 1 >$ $< 0, 0, 5 >$ $< 5, 2, 3 >$
$< 9, 3, 3 >$ $< 4, 1, 5 >$ $< 7, 4, 4 >$ $< 3, 1, 5 >$ $< 12, 4, 1 >$

## Problem 7

The decoded sequence is: *ratatatatat_a_rat_at_a_rat*

# Chapter 8

## Problem 3

According to table 8.3, the optimal stepsize for a laplacian distribution of variance 1 is $\Delta = 0.7309$. The decision boundaries (assuming mean 0) will be $\{-\infty,\ -3\Delta,\ -2\Delta,\ -\Delta,\ 0,\ \Delta,\ 2\Delta,\ 3\Delta,\ \infty\}$ and the reconstruction points will be $\{-7\Delta/2,\ -5\Delta/2,\ -3\Delta/2,\ -\Delta/2,\ \Delta/2,\ 3\Delta/2,\ 5\Delta/2,\ 7\Delta/2\}$.

Now we simply take this quantizer, multiply all values with the input standard deviation $\sqrt{4} = 2$ and add the mean value 3. The best quantizer for the given distribution thus has decision boundaries
$\{-\infty,\ -1.3854,\ 0.0764,\ 1.5382,\ 3,\ 4.4618,\ 5.9236,\ 7.3854,\ \infty\}$
and reconstruction levels
$\{-2.1163,\ -0.6545,\ 0.8073,\ 2.2691,\ 3.7309,\ 5.1927,\ 6.6545,\ 8.1163\}$

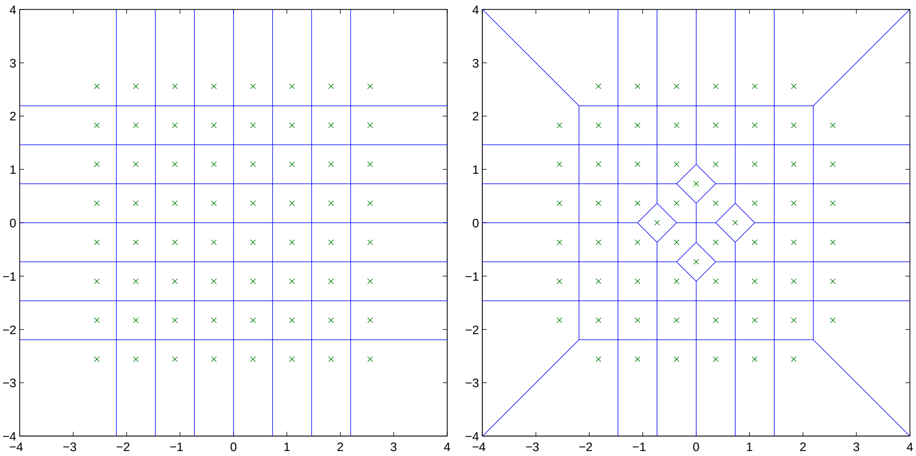
## Problem 6

Uniform quantization gives: -0.5, 1.5, 0.5, 0.5, 3.5, -0.5
Companding plus uniform quantization gives: -0.75, 1.75, 0.25, 0.75, 3.25, -0.25

# Chapter 9

## Problem 1

The reconstruction points and decision regions for the two quantizers look like:



Simulations in Matlab, quantizing two million samples drawn from a Laplace distribution with zero mean and variance 1, gives the distorsions 0.072 and 0.065 respectively. The corresponding SNR-values are 11.4 and 11.9 dB.

## Problem 5

(a) The overload probability is increased by

$$4 \cdot \int_{3\Delta}^{4\Delta} \frac{1}{\sqrt{2}} e^{-\sqrt{2}x} dx \cdot \int_{3\Delta}^{4\Delta} \frac{1}{\sqrt{2}} e^{-\sqrt{2}y} dy + 8 \cdot \int_{3\Delta}^{4\Delta} \frac{1}{\sqrt{2}} e^{-\sqrt{2}x} dx \cdot \int_{2\Delta}^{3\Delta} \frac{1}{\sqrt{2}} e^{-\sqrt{2}y} dy =$$

$$= (e^{-\sqrt{2}\cdot 3\Delta} - e^{-\sqrt{2}\cdot 4\Delta})^2 + 2(e^{-\sqrt{2}\cdot 3\Delta} - e^{-\sqrt{2}\cdot 4\Delta}) \cdot (e^{-\sqrt{2}\cdot 2\Delta} - e^{-\sqrt{2}\cdot 3\Delta}) \approx$$
$$\approx 0.005569$$

(b) The overload probability is decreased by

$$8 \cdot \int_0^{\Delta} \frac{1}{\sqrt{2}} e^{-\sqrt{2}x} dx \cdot \int_{4\Delta}^{5\Delta} \frac{1}{\sqrt{2}} e^{-\sqrt{2}y} dy = 2 \cdot (1 - e^{-\sqrt{2}\cdot\Delta}) \cdot (e^{-\sqrt{2}\cdot 4\Delta} - e^{-\sqrt{2}\cdot 5\Delta}) \approx$$
$$\approx 0.01329$$

# Chapter 10

## Problem 6

Predictor

$$p_{ij} = a \cdot x_{i,j-1} + b \cdot x_{i-1,j}$$

The optimal predictor is given by the solution to

$$\begin{bmatrix} E\{x_{i,j-1}^2\} & E\{x_{i,j-1} \cdot x_{i-1,j}\} \\ E\{x_{i,j-1} \cdot x_{i-1,j}\} & E\{x_{i-1,j}^2\} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} E\{x_{i,j} \cdot x_{i,j-1}\} \\ E\{x_{ij} \cdot x_{i-1,j}\} \end{bmatrix}$$

or, using the auto correlation function $R_{xx}(k,l) = E\{x_{i,j} x_{i+k,j+l}\}$

$$\begin{bmatrix} R_{xx}(0,0) & R_{xx}(1,-1) \\ R_{xx}(1,-1) & R_{xx}(0,0) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} R_{xx}(0,1) \\ R_{xx}(1,0) \end{bmatrix}$$

# Chapter 12

## Problem 1

$$\begin{aligned} |X_1 - X_2|^2 &= (X_1 - X_2)^T (X_1 - X_2) \\ &= (\mathbf{A}^T \Theta_1 - \mathbf{A}^T \Theta_2)^T (\mathbf{A}^T \Theta_1 - \mathbf{A}^T \Theta_2) \\ &= (\Theta_1 - \Theta_2)^T \mathbf{A} \mathbf{A}^T (\Theta_1 - \Theta_2) \\ &= (\Theta_1 - \Theta_2)^T (\Theta_1 - \Theta_2) \\ &= |\Theta_1 - \Theta_2|^2 \end{aligned}$$

## Problem 2

The transform of the first row is

$$(32.5269,\ -1.2815,\ -1.3066,\ 0.4500,\ -1.4142,\ -0.3007,\ 0.5412,\ 0.2549)$$

Slowly varying data gives large magnitudes to the low frequency components.

The transform of the second row is

$$(0.3536,\ 4.2506,\ 0.3499,\ 5.0463,\ 2.4749,\ 8.3837,\ 1.7685,\ 20.3883)$$

Quickly varying data gives large magnitudes to the high frequency components.

Concatenating the two rows, the transform is

$$(23.25,\ 21.78,\ -3.91,\ -6.96,\ -0.68,\ 6.04,\ -3.25,\ -2.57,$$

$$0.75,\ 3.16,\ -6.14,\ 1.61,\ 1.63,\ 5.50,\ -14.24,\ 13.56)$$

High magnitude in both high and low frequency components.

In this case it would be better to use two shorter transforms, for greater compression.

## Problem 3

(a)

$$\frac{1}{4}\begin{bmatrix} 26 & 4 & 8 & 2 \\ 4 & 2 & 2 & 0 \\ 8 & 2 & 4 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix}$$

(b) Same result as in (a).

(c) When doing a separable transform, it doesn't matter if we transform rows or columns first.